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Damage detection using the Lipschitz exponent estimated by the wavelet transform: applications to vibration modes of a beam

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Abstract

Damage detection by the wavelet transform of the fundamental vibration mode receives much attention recently. Many investigations report successful applications of the wavelet transform in damage detection, but most of them appear to lack theoretical justifications. The objective of this contribution is to show the effectiveness of the wavelet transform by means of its capability to estimate the Lipschitz exponent. It is also addressed that the magnitude of the Lipschitz exponent can be used as a useful indicator of the damage extent. As a specific example, damaged beams are investigated both numerically and experimentally. The continuous wavelet transform (CWT) by a Mexican hat wavelet having two vanishing moments is utilized for the estimation of the Lipschitz exponent. The analysis by the CWT also gives a guideline to choose appropriate discrete wavelet transforms. © 2002 Elsevier Science Ltd. All rights reserved.

Keywords: Damage detection; Mode shapes; Lipschitz exponent; Wavelet transform

1. Introduction

Because of practical importance, damage detection has been a subject of intensive investigations. The goal of damage detection is to identify the existence of any flaw and to assess its location and severity. Recently, there has been a growing interest to use modal parameters such as natural frequencies and mode shapes. The advantage of using modal parameters lies in that modal testing is relatively easy. On the other hand, it is not easy to extract local information such as small flaws from modal parameters as they can characterize the global behavior of a structural system quite satisfactorily. Therefore, an accurate, yet efficient diagnostic technique based on modal parameters is very important.

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Most of reported modal parameter-based damage detection methods try to find the correlations between damage and modal parameters. Yuen (1985) proposed a method to utilize eigenfrequencies and eigenmodes by using a finite element model of a cantilever beam. Pandey et al. (1991) attempted to use curvature information by taking the finite difference of displacement mode shapes. By extending the curvature method, Maia et al. (1997) proposed to use the frequency response functions of curvature derived from the frequency response functions of displacement. Pandey and Biswas (1994) also suggested a method to employ an experimentally determined flexibility matrix for damage detection.

Ratcliffe (1997) has recently proposed a method that does not require the mode shape information of an undamaged beam. The idea in Ratcliffe (1997) was to take the Laplacian of the mode shape of a damaged beam. This method is similar to the curvature-based method in that it takes numerical differentiation of measure data, but shown to be more effective in finding small defects in a beam. However, the evaluation of the Laplacian requiring numerical derivatives is prone to significant errors especially when the measured data are noisy.

Another important class of damage detection methods is based on the wavelet transform. The wavelet transform method in damage detection becomes popular because the wavelet transform has localization characteristics and does not require the numerical differentiation of the measure data. Wavelet-based methods have received more attention in dealing with high-frequency wave signals (see, e.g., Jeong and Jang, 2000; Kim and Kim, 2001), but there are also some important applications of wavelets to modal data such as mode shapes.

Liew and Wang (1998) applied the wavelet expansions on the mode shapes obtained analytically for a cracked beam. Deng and Wang (1998) and Wang and Deng (1999) addressed the feasibility of the discrete wavelet transform method for damage detection. Though these works presented quite satisfactory results for damage location identification, no discussion on how to estimate damage extent was provided. No experimental verification was provided in these works, either. Recently, Kim et al. (2000) reported both numerical and experimental results obtained from the application of the discrete wavelet transform on the first fundamental mode shape of a damaged beam. In Kim et al. (2000), the singularity aspect of a damaged beam was investigated, but no thorough theoretical investigation was carried out.

In this work, we present a new method to estimate the damage location and extent in a beam using the fundamental mode shapes. The fundamental mode shape is chosen since it is, in general, most accurately determined by a standard modal testing method. In the present method, the mode shape is wavelet transformed for damage diagnostics, and beam damage is investigated in terms of the Lipschitz exponent—the exponent representing the order of the singularity.

For theoretical investigation, we take the continuous wavelet transform (CWT) of the fundamental mode shape using the Mexican hat wavelet. The basis to choose the Mexican hat wavelet is twofolded: it has two vanishing moments necessary for beam damage investigation, and it also makes the asymptotic analysis of a singularity more effective near small values of the scale parameter. For singularity characterization, the notion of the Lipschitz exponent is utilized, which can be estimated by examining the modulus maximum of the CWT near a singularity.

We illustrate that the modulus maximum accurately determines the damaged location. This discussion is followed by several numerical investigations to link the magnitude of the Lipschitz exponent and the extent of a defect. These numerical findings are confirmed by the actual application of the CWT to the experimental fundamental mode shape of a damaged beam. No mode shape from an undamaged beam may be necessary in the application of the wavelet-based method.

For experimental data that may not be suitable to apply the CWT, discrete wavelet transforms can be also applied. In the application of the discrete wavelet transform, we emphasize the role of vanishing moments in conjunction with the Lipschitz exponent of the mode shape of a damaged beam. The present wavelet application founded by theoretical justification is expected to broaden the applications of wavelets in a wider class of engineering diagnostics.

2. Continuous wavelet transform and the Lipschitz exponent

The CWT is a very useful time–frequency analysis tool with which local features of a signal, such as singularities, can be effectively analyzed. For square-integrable signals $f(x)$, the CWT Wf is defined as (Daubechies, 1992; Mallat, 1998)

$$Wf(u,s) = \int_{-\infty}^{+\infty} f(x)\psi_{u,s}^{*}(x)\,\mathrm{d}x = \int_{-\infty}^{+\infty} f(x)\frac{1}{\sqrt{s}}\psi^{*}\left(\frac{x-u}{s}\right)\mathrm{d}x \tag{1}$$

In (1), $\psi(x)$ is a mother wavelet satisfying the following admissibility condition:

$$\int_{-\infty}^{+\infty} \frac{|\hat{\psi}(\omega)|^2}{|\omega|}\,\mathrm{d}\omega < \infty \tag{2}$$

where $\hat{\psi}(\omega)$ is the Fourier transform of $\psi(x)$. The existence of the integral in (2) requires that

$$\hat{\psi}(0) = 0, \quad \text{i.e.,} \int_{-\infty}^{+\infty} \psi(x)\,\mathrm{d}x = 0 \tag{3}$$

The function $\psi_{u,s}(x)$ is dilated by the scaling parameter s and translated by the translation parameter u of the mother wavelet $\psi(x)$.

An important property of the CWT is the ability to characterize the local regularity of functions. The local regularity is often measured by the Lipschitz exponent. The Lipschitz exponent α may be defined as the following.

A function f is said to be Lipschitz $\alpha \geqslant 0$ at $x = v$ if there exists $K > 0$ and a polynomial p_v of degree m (m is the largest integer satisfying $m \leqslant \alpha$) such that

$$f(x) = p_v(x) + \varepsilon_v(x) \tag{4}$$

$$|\varepsilon_v(x)| \leqslant K|x - v|^{\alpha} \tag{5}$$

For instance, a function is not differentiable at $x = v$ if $0 < \alpha < 1$. Therefore, the Lipschitz exponent α characterizes the nature of singularity at $x = v$.

When a structure has a flaw or defect, the vibration mode shape changes depending on the location and type of damage. Small defects are difficult to identify, but they introduce some sorts of singularities to the vibration mode shapes. These singularities may be characterized by the concept of the Lipschitz exponent discussed briefly above. Therefore, we need an analysis tool to estimate the Lipschitz exponent from vibration mode shapes of damaged structures; this is the objective of the present investigation. As an effective estimator of the Lipschitz exponent, we will utilize the CWT.

In applying the wavelet transform for the Lipschitz exponent estimation, the notion of the vanishing moment plays an important role. A wavelet $\psi(x)$ is said to have n vanishing moments if it satisfies the following conditions:

$$\int_{-\infty}^{+\infty} x^k \psi(x)\,\mathrm{d}x = 0 \quad \text{for } 0 \leqslant k < n \tag{6}$$

The condition (6) states that the wavelet having n vanishing moments is orthogonal to polynomials of up to degree $n - 1$.

If the CWT with $n \geqslant \alpha$ is applied to the function of the form (4), the result becomes

$$Wf(u,s) = W\varepsilon_v(u,s) \tag{7}$$

since $Wp_v(u,s) = 0$. Because of the vanishing moment property of the wavelet (with sufficiently large n such that $n \geqslant \alpha$), the CWT focuses only on a singular part of a function.

Jaffard (1991) shows that if a square-integrable function $f(x)$ is Lipschitz $\alpha \leqslant n$ at $x = v$, then the asymptotic behavior of the CWT Wf near $x = v$ becomes (see also Mallat (1998))

$$|Wf(u,s)| \leqslant A' s^{\alpha+(1/2)} \left(1 + \left|\frac{u-v}{s}\right|^{\alpha}\right) \quad (A' > 0) \tag{8}$$

Near the cone of influence of $x = v$, (8) reduces to

$$|Wf(u,s)| \leqslant A s^{\alpha+(1/2)} \tag{9}$$

The high-amplitude wavelet coefficients are in the cone of the influence of the singularity. It is more convenient to use the following form, equivalent to (9), in the estimation of the Lipschitz exponent:

$$\log_2 |Wf(u,s)| \leqslant \log_2 A + (\alpha + \tfrac{1}{2}) \log_2 s \tag{10}$$

In the actual application of (9), the locus of the local modulus maximum of the wavelet transform should be determined. This will be illustrated later with numerical examples.

3. Wavelet selection for damaged beam analysis

The specific application of the CWT for the Lipschitz-exponent estimation will be made in a beam. A beam is the simplest structural element, yet quite important one. A model beam is shown in Fig. 1 where the beam has a defect represented by an abrupt thickness change. In this section, we derive the bound of the Lipschitz exponent using the simplest Euler beam theory. Based on the estimated Lipschitz exponent, we suggest an optimal wavelet for the estimation of the Lipschitz exponent in the present problem.

The rigorous analysis of the Lipschitz exponent may have to be carried out by the exact elasticity theory. However, the experimental procedure to measure vibration modes may be more consistent with a beam theory. This is because modal data are acquired only at a few discrete points on the top (or) bottom surface of a beam. Therefore, the estimation of the singularity due to a small crack can be achieved quite satisfactorily by the beam theory. In this work, we use the simplest Euler beam theory.

When the Euler beam theory is used, the field variables are displacement, rotation, moment and shear force. If one denotes by $x = v$ the point of an abrupt thickness change in a beam, one can write the continuity condition across the discontinuity as

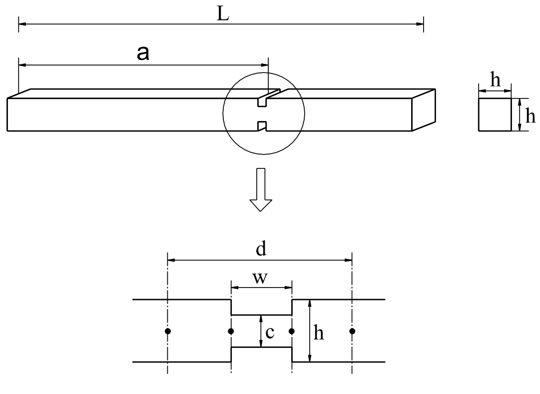

Fig. 1. A model for a damaged beam.

$$\text{Displacement:}\quad w(v^+) = w(v^-) \tag{11a}$$

$$\text{Rotation:}\quad \frac{\mathrm{d}w(v^+)}{\mathrm{d}x} = \frac{\mathrm{d}w(v^-)}{\mathrm{d}x} \tag{11b}$$

$$\text{Moment:}\quad EI(v^+)\frac{\mathrm{d}^2w(v^+)}{\mathrm{d}x^2} = EI(v^-)\frac{\mathrm{d}^2w(v^-)}{\mathrm{d}x^2} \tag{11c}$$

$$\text{Shear force:}\quad EI(v^+)\frac{\mathrm{d}^3w(v^+)}{\mathrm{d}x^3} = EI(v^-)\frac{\mathrm{d}^3w(v^-)}{\mathrm{d}x^3} \tag{11d}$$

where the subscripts + and − are used to denote the quantities just at the right and left of the discontinuous point. For the case when an actual defect has a finite dimension, the same analysis can be applied to the discontinuities at each end of the defect.

Since the bending rigidity EI is related to the beam thickness as in

$$EI = \frac{E \times (\text{width}) \times (\text{thickness})^3}{12} \qquad (E: \text{Young's modulus})$$

EI is discontinuous across the discontinuity, i.e.,

$$EI(v^+) \neq EI(v^-) \tag{12}$$

Using (11a)–(11d) and (12), one can conclude that only the lateral displacement $w(x)$ and its first derivative are continuous. Therefore, the Lipschitz exponent of the mode shapes of a damaged beam must lie between 1 and 2:

$$1 < \alpha < 2 \tag{13}$$

Based on the finding (13), we conclude that to extract the Lipschitz exponent in the present beam problem using the wavelet transform, the minimum number of the vanishing moments is $n = 2$:

$$n \geqslant 2 \tag{14}$$

Regardless of the CWT or the discrete wavelet transform, the condition (14) must be fulfilled. Since wavelets with more vanishing moments have longer support sizes, a tradeoff between vanishing moments and support sizes should be considered. However, we adopt the wavelet with $n = 2$ for the best localization property.

Another important factor to consider in the wavelet selection is the continuity of the modulus maximum of $Wf(u,s)$. If a wavelet $\psi(x)$ is the nth derivative of a Gaussian $\theta(x)$ as in $\psi(x) = (-1)^n(\mathrm{d}^n\theta(x)/\mathrm{d}x^n)$, the modulus maximum of $Wf(u,s)$ belongs to a connected curve that is never interrupted for decreased scales (Mallat, 1998). Furthermore, the corresponding wavelet has n vanishing moments. Based on these observations, we propose to employ the following wavelet for the present problems:

$$\psi(x) = \frac{\mathrm{d}^2\theta(x)}{\mathrm{d}x^2} \tag{15}$$

The wavelet in (15) is usually referred to as the Mexican hat wavelet and has the following explicit expression:

$$\psi(x) = \frac{2}{\sqrt{3\sigma}}\pi^{-1/4}\left(\frac{x^2}{\sigma^2} - 1\right)\exp\left(\frac{-x^2}{2\sigma^2}\right) \tag{16}$$

The graph $-\psi(t)$ for $\sigma = 1$ is shown in Fig. 2.

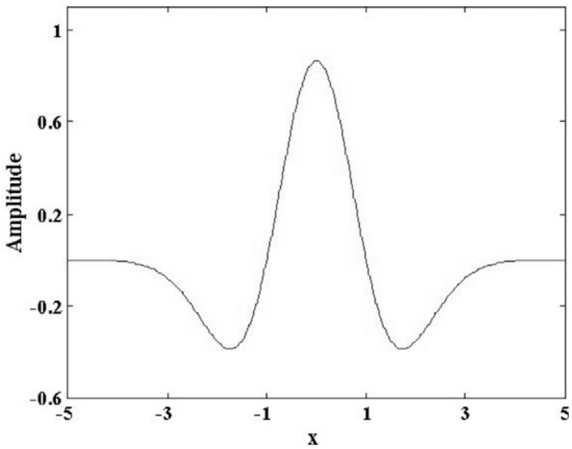

Fig. 2. The graph $-\psi(x)$ with $\sigma = 1$.

4. Simulations for the Lipschitz exponents

Before applying the CWT to experimental mode shapes, it is very useful to perform numerical simulations. The objective of this section is to study:

1. the relation between the damage extent and the Lipschitz exponent;
2. the comparison of the exponent estimation for mode shapes for different modal numbers;
3. the effect of the sampling distance on the exponent.

The simulation results for the effects of noise on the Lipschitz exponent will be presented in parallel with experimental results in the next section.

For numerical simulations, the beam model in Fig. 1 is the target. A defect, or a flaw, in the beam is realized by an abrupt thickness change because this configuration is easy for both numerical and experimental studies. Numerical modal analysis is carried out by a commercial finite element program (ANSYS, 1993). The default width and thickness of the beam is $h = 20$ mm, but the height c of the damaged part may vary from 2 to 14 mm. Unless stated otherwise, the damage size w will be taken as $w = 0.5$ mm. The beam has free ends.

In using ANSYS, 2400 two-node plane beam elements (element size = 0.5 mm) are used. The number of elements used for the present problem is more than sufficient to obtain converged results. The material data used are: Young's modulus $E = 70$ GPa, density $\rho = 2700$ kg/m^3.

As the first case study, we consider a damaged beam with $c = 6$ mm, i.e., $c/h = 0.3$. The first bending mode shapes of a damaged and undamaged beam are compared in Fig. 3(a). There is only a slight difference between the two mode shapes as expected. Now we take the wavelet transform on the first mode shape of the damaged beam using the Mexican hat wavelet in (16). The magnitude $|Wf(u,s)|$ is plotted in the scale-translation plane. The horizontal axis represents the translation parameter u, which denotes the distance from the origin of the x coordinate. Here, the mode shape is denoted by $f(x)$.

The modulus maximum lying inside of the cone of influence is clearly seen in Fig. 3(b), and the application of a ridge algorithm (Mallat, 1998) gives the result shown in Fig. 3(c). Observe that the limit of the modulus maximum line approaches $u \approx 800$ mm, where the damage (of size $w = 0.5$ mm) is located as the scale parameter s goes to 1. In the present simulation, the element size is intentionally chosen to be the same as the damage size w. This selection is made because in most of actual experiments, a sampling distance

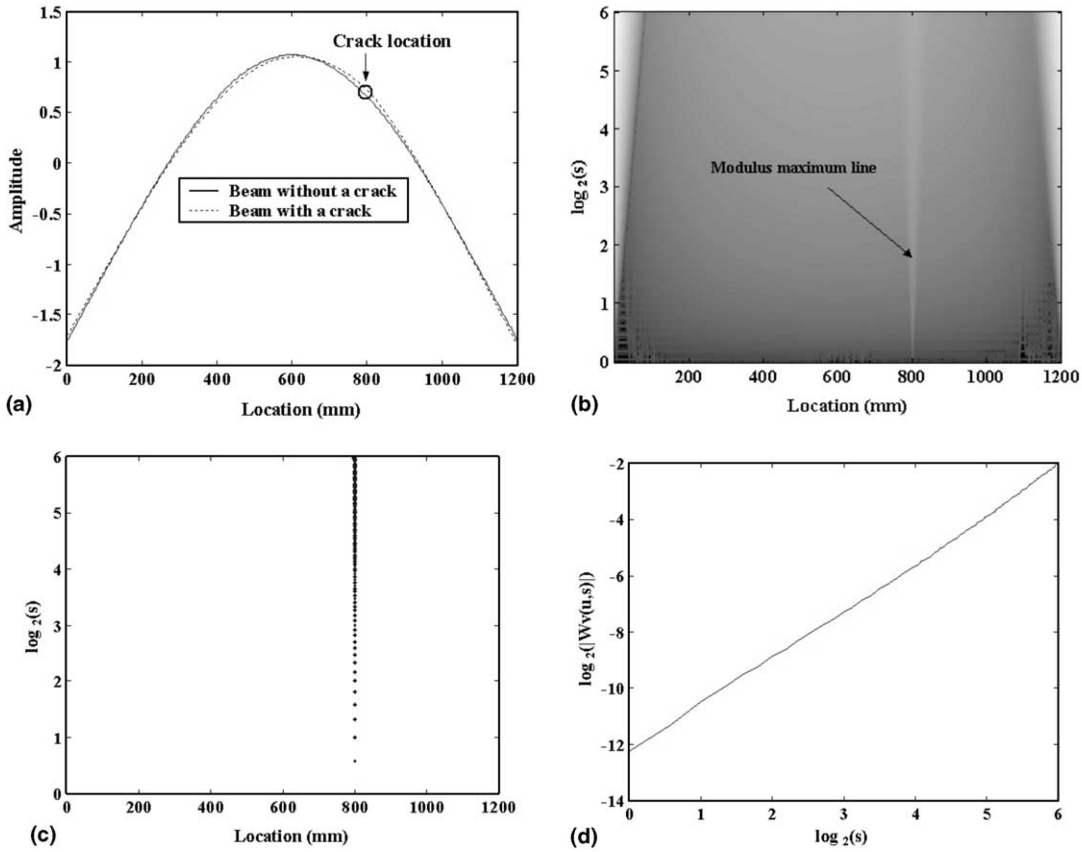

Fig. 3. Simulation for a damaged beam with $c/h = 0.3$. The damage location is $a = 800$ mm. (a) The first bending mode shape denoted by $f(x)$, (b) the contour plot of $|Wf(u,s)|$, (c) the locus of the modulus maximum, (d) the decay behavior along the modulus maximum line.

exceeds the damage size. If the element size is much smaller than the damage size, however, two modulus maximum will appear. This is not pursued here to simulate actual experimental situations.

To estimate the Lipschitz exponent, the decay behavior along the modulus maximum line shown in Fig. 3(b) is plotted in Fig. 3(d). By applying the well-known linear regression technique, one can estimate the Lipschitz exponent α as

$$\alpha = 1.31$$

This value indeed lies between the lower and upper bounds determined as in (13).

To investigate the relation between the damage extent and the Lipschitz exponent, we vary the damage size from $c/h = 0.1$ to 0.7. The damage location is fixed at $a = 800$ mm. Repeating the same procedure used in the first case study, the results shown in Fig. 4 are obtained. It is clear that the results of Fig. 4 are consistent with the physical behavior: as the damage extent becomes severer (meaning that c/h becomes smaller), the singularity gets worse (meaning that the Lipschitz exponent becomes smaller). This suggests a possibility to correlate the damage extent to the Lipschitz exponent. However, it is remarked that noise in measured signals, impossible to avoid in actual experimentation, lowers the values of the Lipschitz exponent. This will be discussed in the next section in more details.

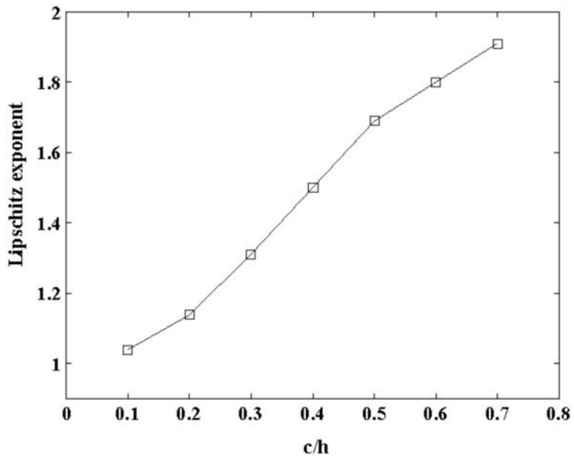

Fig. 4. The estimates of the Lipschitz exponent for different values of c/h.

We also address that the same amount of the damage extent at different locations may yield different estimates of the Lipschitz exponent. To show this, we carry out the Lipschitz exponent analysis for varying damage locations. Again, the fundamental bending mode is wavelet transformed. Table 1 shows the values of the Lipschitz exponent for three different damage locations. The thickness of the damaged segment varies from $c/h = 0.1$ to 0.7 as before. Depending on the damage locations, slight changes in the exponent values appear. This is because the curvature of the mode shape (of an undamaged beam) at different locations affects the local singularity behavior as the analysis is carried out in a finite-dimensional space of (u, s).

The effects of the variation rate of the mode shape may be understood more clearly by comparing the exponents estimated from higher mode shapes. To this end, we attempt to estimate the Lipschitz exponent from the first three bending mode shapes. The damage location is $a = 800$ mm, and the width is $w = 0.5$ mm as before. For varying values of c/h, the values of the Lipschitz exponents are plotted in Fig. 5. The higher the mode number is, the smaller the exponent value is. If a function is rapidly varying, but is not sampled at sufficiently many locations, the sampled version of the function will look more singular. The present result indeed reveals why the first mode shape is preferred for damage detection.

It is worth examining how sampling distances for modal testing affects the value of the Lipschitz exponent. The sampling distance is denoted by d in Fig. 1 and the sampling points are regarded as actual sensing points. The model in consideration is exactly the same beam used in the first case ($L = 1200$ mm, $a = 800$ mm, $w = 0.5$ mm). Note that regardless of the sampling distance, the finite element analysis is carried out with 2400 beam elements.

The values of the Lipschitz exponent are plotted in Fig. 6 for varying sampling distances. The singularity estimation is strongly influenced by the sampling distance. If the number of sampling points decreases, i.e., the sampling distance increases, the value of the Lipschitz exponent increases. This means that the use of

Table 1
The estimates of the Lipschitz exponent α for different damage locations at $x = a$

a (mm)	Estimates of α						
	$c/h = 0.1$	0.2	0.3	0.4	0.5	0.6	0.7
600	1.04	1.17	1.38	1.50	1.70	1.78	1.83
800	1.04	1.14	1.31	1.50	1.69	1.80	1.91
1000	1.05	1.15	1.30	1.51	1.69	1.86	1.75

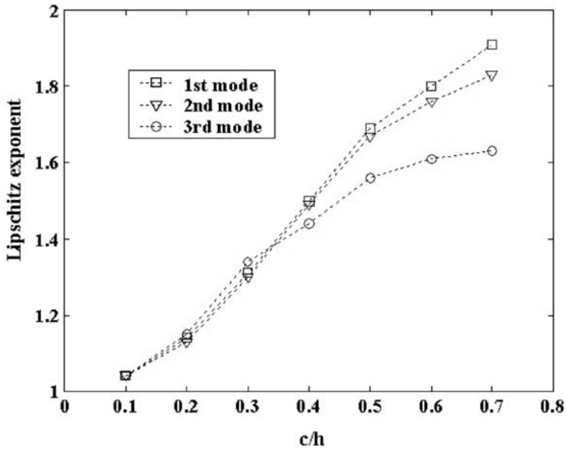

Fig. 5. The estimates of the Lipschitz exponent for three different mode shapes.

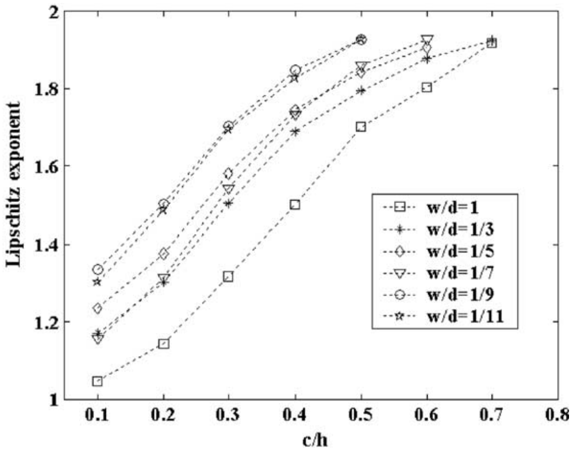

Fig. 6. The effects of the sampling distances on the estimates of the Lipschitz exponent (w = damage size, d = sampling distance).

insufficient sampling points loses information on the correct local singular behavior. In practical modal testing where sampling resolution is not high, some care should be taken for the correct interpretation of the meaning of the exponent values. However, Fig. 6 suggests that not only the presence but also the extent of a defect can be diagnosed quite satisfactorily by the present method (observe that the analysis even with $w/d = 1/11$ characterizes the correct behavior).

5. Experiments for the Lipschitz exponent evaluation

To validate the present method, an experiment has been performed. The shape of the test beam is the same as the one in Fig. 1, and the numerical data are $L = 1200$ mm, $h = 20$ mm, $a = 810$ mm, $w = 30$ mm, $c = 10$ mm ($c/h = 0.5$). The sampling distance is $d = 30$ mm, and thus data are acquired at 39 points.

Fig. 7(a) shows the first bending mode shape of the damaged beam, which is determined by modal testing. Small circles in the figure represent the measurement points whereas the big circle indicates the location of the damage. The experimental equipment consists of an accelerometer (B&K model-4375), a preamplifier (B&K model-2626), an FFT analyzer (HP-35670A), and modal analysis software (IDEAS-TDAS). To reduce noise, measured signals were averaged five times in the frequency domain.

The contour plot for the magnitude $|Wf(u,s)|$ of the experimental mode shape $f(x)$ is presented in Fig. 7(b). The thick line represents the modulus maximum line arriving at $820 \leqslant u \leqslant 830$ mm, which corresponds to a point near the center of the damaged part. As pointed out earlier, if a sufficiently large number of measurement points was used, two modulus maximum lines would appear. In the present case with insufficient measurement points, they merge together. However, the singularity nature does not disappear.

The decay behavior of $|Wf(u,s)|$ along the modulus maximum line shown in Fig. 7(b) is plotted in Fig. 7(c). Carrying out a linear regression analysis on the result shown in Fig. 7(c) give an estimate of the Lipschitz exponent as

$$\alpha_{\exp} = 1.31 \tag{17}$$

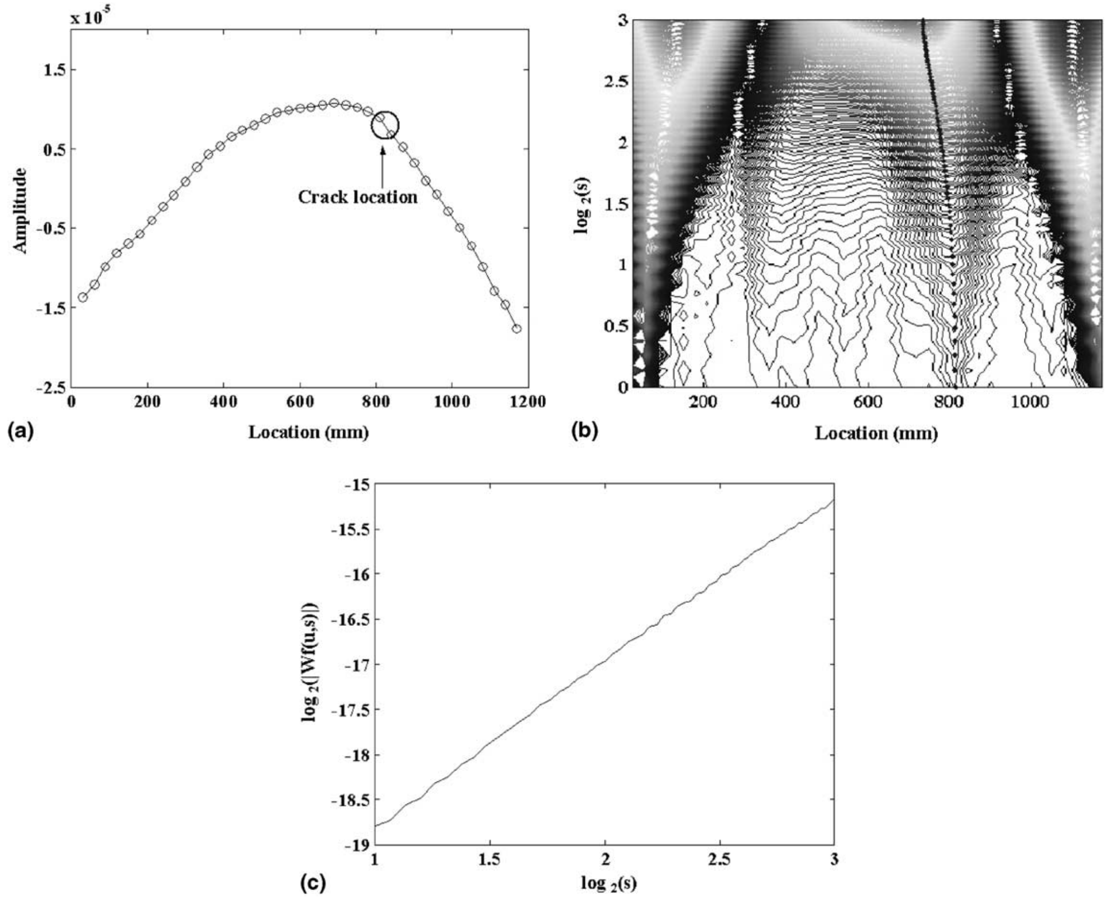

Fig. 7. The wavelet and Lipschitz singularity analysis performed on the experimental mode shape of a damaged beam. (a) The first mode shape $f(x)$ of a damaged beam at $a = 800$ mm, (b) the contour plot of $|Wf(u,s)|$ with a modulus maximum emanating from near $u = 825$ mm, (c) the decay behavior along the modulus maximum line.

To compare the experimental Lipschitz value with the numerical Lipschitz value, a finite element analysis is performed for the test model. Though the finite element analysis was done with 2400 elements, the mode shape is sampled only at the measurement points for realistic comparison. The numerical mode shape and the corresponding decay behavior are shown in Fig. 8(a) and (b), respectively. The estimated value of the exponent is

$$\alpha_{\text{num}} = 1.51 \tag{18}$$

Comparing (17) and (18), one can see that the experimental estimation of the exponent predicts stronger singularity (meaning that $\alpha_{\text{exp}} < \alpha_{\text{num}}$) than it actually is. This discrepancy may be due to noise introduced in the measurement process.

To investigate the effect of noise on the estimated value of the Lipschitz exponent, we return to the numerical mode shown in Fig. 8(a) and add some white noise. To this end, we consider the following model:

$$\bar{f}(x) = f(x) + n(x), \quad n(x) \sim N(0, \sigma_{\text{n}}^2) \tag{19}$$

Here, $\bar{f}(x)$ is a contaminated signal of $f(x)$ by a Gaussian noise $n(x) \sim N(0, \sigma_{\text{n}}^2)$ where σ_{n} denotes the standard deviation of a Gaussian noise.

Fig. 9(a) compares $f(x)$ and $\bar{f}(x)$ with $\sigma_{\text{n}} = 0.04$. These two functions are denoted by a "clean mode" and a "noisy mode", respectively where $f(x)$ represents the numerical mode shape plotted in Fig. 8(a). The noisy numerical mode shape appears quite close to the experimental mode shape. The decay behavior for $\bar{f}(x)$ is also shown in Fig. 9(b). The estimated Lipschitz exponent is

$$\alpha_{\text{num}} \quad (\text{with } \sigma_{\text{n}} = 0.04 \text{ or } (N/S)_{\text{n}} = 0.04) = 1.41 \tag{20}$$

When the same analysis is repeated with $\sigma_{\text{n}} = 0.05$, the following result is obtained.

$$\alpha_{\text{num}} \quad (\text{with } \sigma_{\text{n}} = 0.05 \text{ or } (N/S)_{\text{n}} = 0.06) = 1.25 \tag{21}$$

where the symbol $(N/S)_{\text{n}}$ is the noise-to-signal ratio defined as

$$(N/S)_{\text{n}} = \frac{\left\| \bar{f}(x) - f(x) \right\|}{\| f(x) \|} \tag{22}$$

If the added noise to the experimental mode shape is assumed to be a Gaussian noise, $(N/S)_{\text{n}}$ should be somewhere between 0.04 and 0.06. (However, the estimation of $(N/S)_{\text{n}}$ is not the purpose of this work.) The

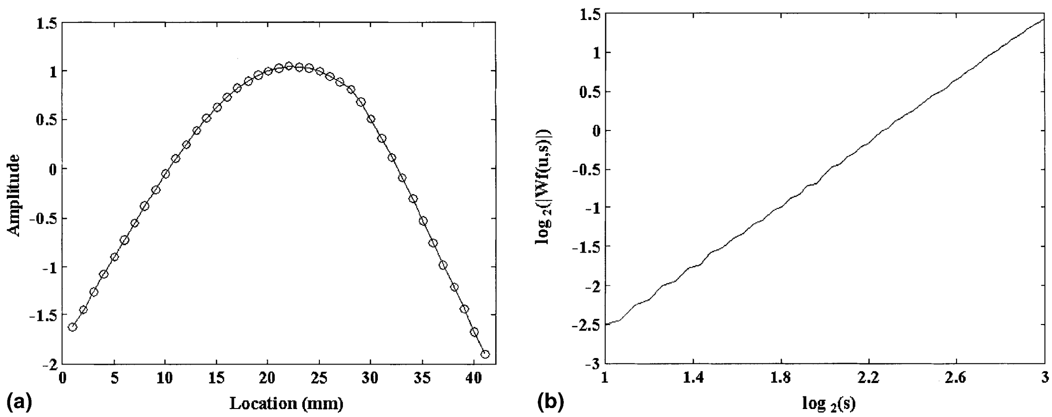

Fig. 8. Analysis of the simulated mode shape sampled exactly at the same points as the experimental sampling points. (a) The numerical mode shape (sampling points marked by circles), (b) the decay behavior along the modulus maximum line.

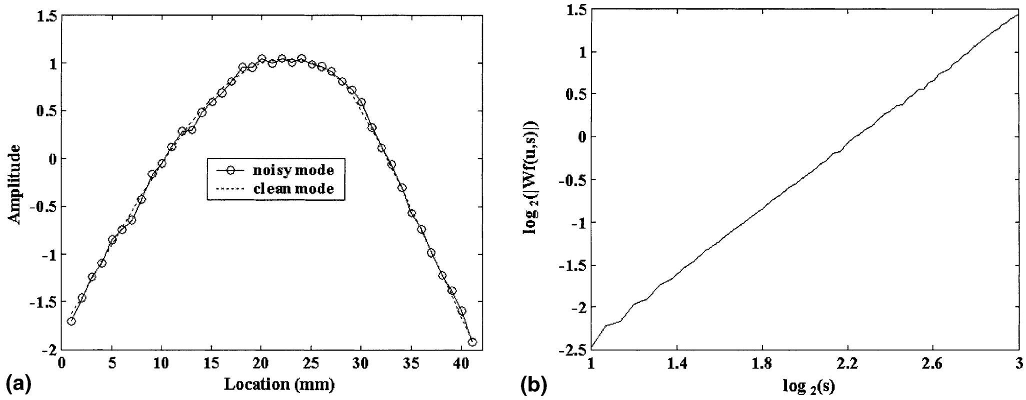

Fig. 9. Simulation for the contaminated signal with the noise level $\sigma_n = 0.04$. (a) The comparison of the noisy and clean mode shapes, (b) the decay behavior along the modulus maximum line.

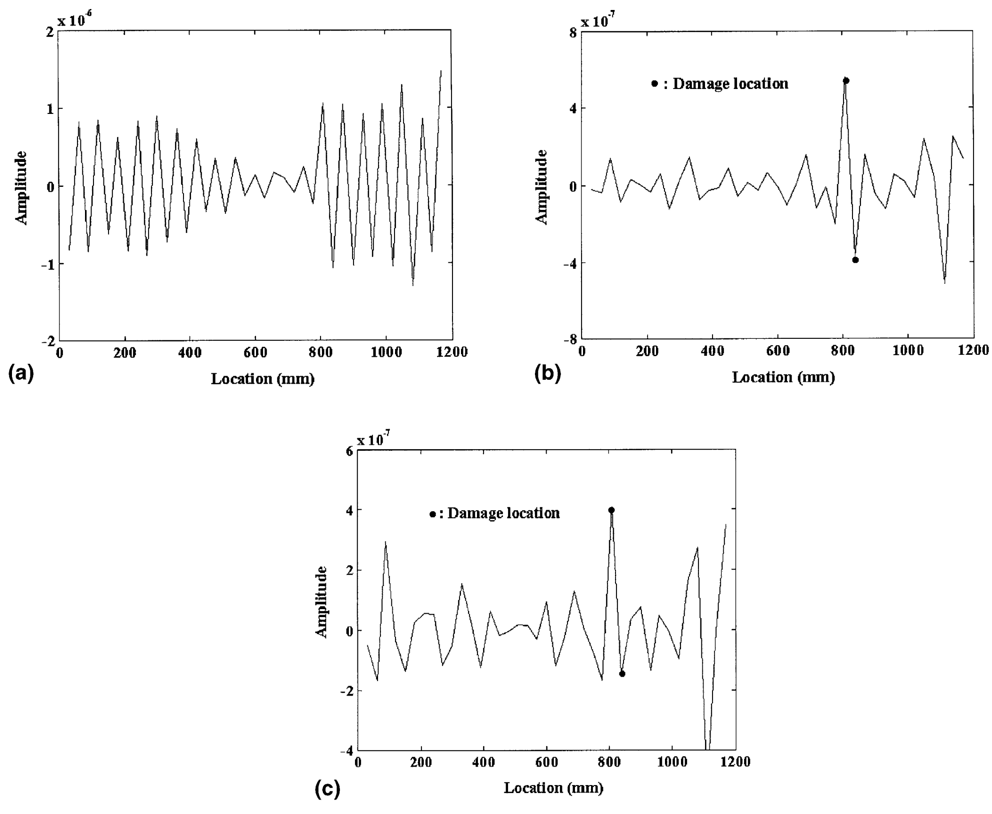

Fig. 10. The discrete wavelet transform performed on the experimental mode shape of a damaged beam. (a) The first-level wavelet coefficients by the Daubechies wavelets having one vanishing moment, (b) the first-level wavelet coefficients by the Daubechies wavelets having two vanishing moments, (c) the first-level wavelet coefficients by the Daubechies wavelets having three vanishing moments.

point we want to address is that the noise added during the actual experimentation reduces the value of the Lipschitz exponent because it makes the mode shape more singular. Therefore, the estimate of the Lipschitz exponent using experimental data always gives a lower value than its true value.

At this point, it is worth applying the discrete wavelet transform to experimental mode shapes (Kim et al., 2000). One advantage of the discrete wavelet transform over the CWT in damage detection is efficiency. The discrete wavelet transform is performed by a fast algorithm that cascades discrete convolutions with low and high pass filters and subsamples the output (Mallat, 1998).

The discrete wavelet transform has been applied to the experimental modal shape. Fig. 10(a)–(c) plot the first-level wavelet coefficients by the Daubechies wavelets having (a) one, (b) two and (c) three vanishing moments (Daubechies (1992) for the detailed accounts of the Daubechies wavelets). Earlier in Section 3, we have shown that the minimum number of vanishing moments to detect the present damage is two. As expected, the presence of the damage in a beam cannot be identified in Fig. 10(a) where the Daubechies wavelet with one vanishing moment. However, wavelets with two or more vanishing moments are capable of detecting the damage (note that the large amplitudes at both ends are due to the boundary distortion).

We make a few more remarks on the discrete wavelet transform. Even when the CWT experiences some difficulties because of low measurement resolution, the discrete wavelet transform provides some useful intuition on the existence of a defect. In this case, however, wavelets having more vanishing moments than theoretically required may be needed to yield satisfactory results.

6. Conclusions

This investigation gives some theoretical foundations for the effectiveness of the wavelet transform applied to mode shapes. The present discussion was given in terms of the wavelet capability to estimate the Lipschitz singularity. The following is the summary of the findings in this investigation.

1. The CWT by the Mexican hat wavelet provides the necessary information to estimate the Lipschitz exponent.
2. A damage beam has the Lipschitz exponent lying between 1 and 2.
3. The first fundamental mode is more useful than higher modes in the exponent estimation.
4. The damage size is well correlated with the magnitude of the Lipschitz exponent.
5. Larger sampling distances give higher exponent values, that is, they lessen the apparent strength of singularity. However, the correlation between the damage extent and the exponent value is preserved even for relatively large sampling distances.
6. Noise added to experimental data reduces the Lipschitz exponent value; noise increases the strength of the singularity.
7. The minimum number of the vanishing moment in the application of the discrete wavelet transform can be predicted by the Lipschitz exponent for the damaged beam.

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